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Side-wall boundary layers in rotating axial flow

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The boundary layers forming on the walls of an aligned cylinder in a rotating fluid in axial motion are studied theoretically. The analysis shows that the side-wall boundary layer is of the Blasius type when the Rossby number exceeds the inverse square root of the Reynolds number and is transformed to the Stewartson $\frac{1}{3}$-layer when the Rossby number is less than this value. A second thicker boundary layer is superimposed on the $\frac{1}{3}$-layer whenever the difference between the azimuthal velocities of the ambient fluid and the boundary exceeds the axial velocity. Its thickness varies according to the relative magnitudes of these velocities and yields the Stewartson $\frac{1}{4}$-layer thickness only when the ratio of the azimuthal velocity difference to the axial velocity is of order $E^{\frac{1}{4}}$, where E is the Ekman number. A uniformly valid solution is obtained for the first case when the boundary layer is of the Blasius type.

1. Introduction

Boundary or shear layers in the vicinity of discontinuity surfaces which are parallel to the axis in contained rotating flows are known to have a sandwich structure, which was first derived analytically by Stewartson (1957) and shown to exist experimentally by Baker (1967). They consist of an outer layer of thickness $E^{\frac{1}{4}}$ and an inner layer of thickness $E^{\frac{1}{3}}$, where E is the Eckman number based on the length of the layer. These characteristic scales are determined theoretically by the fact that, in a rotating cylinder, for example, the side-wall layers must match the Eckman layers on the end walls. They are, therefore, intimately dependent on the existence of an Ekman layer which induces at most an axial velocity of order $E^{\frac{1}{2}}$. On the other hand, if axial motions which are much larger than any rotational velocity are imposed on a rotating fluid, one would expect to find a Blasius type of layer in place of the double or, depending on the boundary conditions, single-layer structure found by Stewartson. The following question then arises. What is the correct boundary-layer structure for prescribed values of the axial and angular velocities and what intermediate structures, if any, appear in the transition from the Blasius layer to the Stewartson layers? This problem, which arose initially from a desire to construct a continuous-axial-flow rotating water tunnel, is investigated analytically using a careful scaling analysis and the techniques of singular perturbation methods.

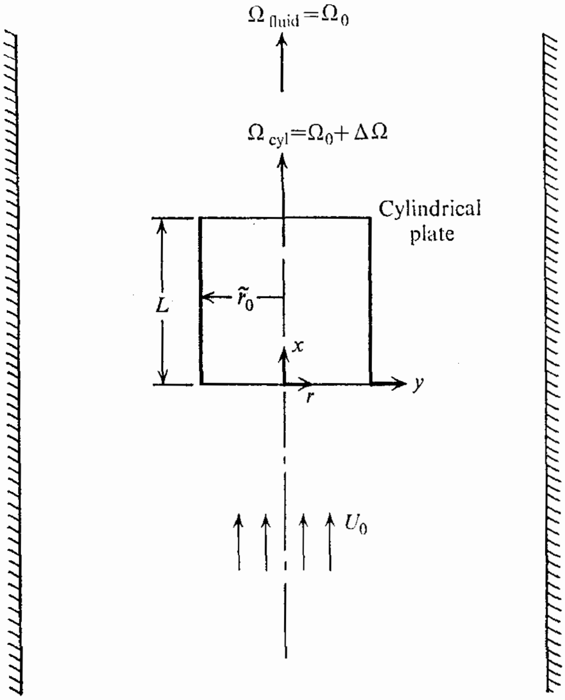

FIGURE 1. A schematic diagram of the flow model.

2. Formulation

We consider the uniform axial flow of an incompressible rotating fluid through a slice of a thin-walled hollow cylinder of radius $\tilde{r}_0$ and length L whose axis coincides with the axis of rotation. The cylinder is taken to be fixed axially, but rotates about its axis at an angular speed different from that of the ambient fluid. The radius of the fluid container is assumed to be much larger than that of the hollow cylinder. We denote the axial velocity of the fluid by U_0, its angular velocity by Ω_0, and the angular velocity of the cylinder by $\Omega_0+\Delta\Omega$ (see figure 1). The dimensionless equations of motion for the fluid, written with respect to a cylindrical polar co-ordinate system rotating with speed Ω_0 and assuming axial symmetry, are

$$u_x+w_r+w/r = 0, \tag{2.1}$$

$$uu_x+wu_r = -p_x+(1/R)\nabla^2 u, \tag{2.2}$$

$$uv_x+wv_r+\frac{vw}{r}+\frac{2}{Ro}w = \frac{1}{R}\left(\nabla^2-\frac{1}{r^2}\right)v \tag{2.3}$$

and

$$uw_x+ww_r-\frac{v^2}{r}-\frac{2}{Ro}v = -p_r+\frac{1}{R}\left(\nabla^2-\frac{1}{r^2}\right)w, \tag{2.4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}. \tag{2.5}$$

All velocities have been scaled with U_0 and all lengths with L. The azimuthal velocity is purposely not scaled with the relative azimuthal velocity so that the limiting case of vanishing $\Delta\Omega$ can be discussed without difficulty. The two parameters appearing are the Reynolds number ($R = U_0L/\nu$) and the Rossby number ($Ro = U_0/\Omega_0 L$). Defining a stream function ψ by

$$u = \frac{1}{r}\frac{\partial}{\partial r}(r\psi), \quad w = -\frac{1}{r}\frac{\partial}{\partial x}(r\psi), \tag{2.6}$$

and introducing the co-ordinate transformation

$$r = r_0 + y, \quad r_0 = \tilde{r}_0/L, \tag{2.7}$$

so that y measures distance radially from the surface of the cylinder, the equations of motion reduce to the coupled set

$$\begin{aligned}\left[L(x,y;\psi) - \frac{1}{R}\nabla_1^2\right]\nabla_1^2\psi + \frac{2}{Ro}v_x &= -\frac{1}{r_0+y}\{L(x,y;\psi)\,\psi_r + (\psi\nabla_1^2\psi + v^2)_x\} \\ &\quad + \frac{1}{(r_0+y)^2}\{L(x,y;\psi)\,\psi - 2\psi_x\psi_y - \psi\psi_{xy}\} + \frac{4\psi\psi_x}{(r_0+y)^3} \\ &\quad + \frac{1}{R}\left\{\frac{2\nabla_1^2\psi_y}{r_0+y} - \frac{2}{(r_0+y)^2}(\nabla_1^2\psi + \psi_{yy}) + \frac{4\psi_y}{(r_0+y)^3} - \frac{3\psi}{(r_0+y)^4}\right\}\end{aligned} \tag{2.8}$$

and
$$\left[L(x,y;\psi) - \frac{1}{R}\nabla_1^2\right]v - \frac{2}{Ro}\psi_x = \frac{1}{r_0+y}\left\{\psi_x v - \psi v_x + \frac{1}{R}\left(v_y - \frac{v}{r_0+y}\right)\right\}, \tag{2.9}$$

where
$$L(x,y;\psi) = \psi_y\,\partial/\partial x - \psi_x\,\partial/\partial y, \tag{2.10}$$

$$\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2. \tag{2.11}$$

Equations (2.8) and (2.9) are the azimuthal components of the vorticity and momentum equations, respectively, and are written so that the right-hand sides contain all terms accounting for the transverse curvature of the cylinder. They are an exact coupled set of equations and any true boundary-layer scaling must be consistent with both of them simultaneously.

The boundary conditions for the problem are the no-slip conditions on the hollow cylinder surface,

$$\psi(x,0) = \psi_y(x,0) = 0, \quad v(x,0) = (\tilde{r}_0|\Delta\Omega|/U_0)\,\mathrm{sgn}\,\Delta\Omega, \tag{2.12a}$$

and similar conditions on the surface of the fluid container. However, if the diameter of the containing side walls is much larger than the diameter of the hollow cylinder and also larger than the length of the hollow cylinder, the container can be assumed to be infinitely large as far as the boundary layer on the cylinder is concerned. In what follows, these conditions are assumed to be satisfied, so that the boundary conditions on the outer fluid container are replaced by

$$\lim_{y\to\infty}\psi(x,y) = \tfrac{1}{2}(r_0+y), \quad \lim_{y\to\infty}v(x,y) = 0. \tag{2.12b}$$

The ratio of the difference between the azimuthal velocities of the hollow cylinder and the ambient fluid to the imposed axial velocity, which appears in the second of (2.12a), plays an important role in the analysis and will subsequently be denoted by

$$\delta = \tilde{r}_0|\Delta\Omega|/U_0. \tag{2.13}$$

3. Boundary-layer scaling and equations

The flow behaviour in the immediate vicinity of the cylinder is investigated assuming that the influence of viscosity is limited to radially narrow regions. The conditions for the validity of this assumption are to be determined in the course of the analysis. The (boundary-layer) assumption is made explicit by the transformations

$$\zeta = y/\epsilon \quad (\epsilon \ll 1), \tag{3.1}$$

$$\psi(x,y) = \epsilon\Psi(x,\zeta) \tag{3.2}$$

and

$$v(x,y) = \sigma V(x,\zeta), \tag{3.3}$$

where ϵ and σ are undetermined functions of the parameters R, Ro and δ. The stream function transformation is dictated by matching requirements for the axial velocity, and the transformation of the azimuthal velocity is required to maintain the proper scaling in the momentum equation (2.9) and the boundary condition (2.12*a*). The parameter ϵ is determined by requiring the coefficient of the largest viscous term in the vorticity equation (2.8) to be unity and all other terms to be of order unity or smaller.

The transformed vorticity and momentum equations are

$$\left[L(x,\zeta;\Psi)-\frac{1}{\epsilon^2 R}\left(\epsilon^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial\zeta^2}\right)\right]\left(\epsilon^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial\zeta^2}\right)\Psi+\frac{2\epsilon\sigma}{Ro}V_x = 0 \tag{3.4}$$

and

$$\left[L(x,\zeta;\Psi)-\frac{1}{\epsilon^2 R}\left(\epsilon^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial\zeta^2}\right)\right]V-\frac{2\epsilon}{\sigma Ro}\Psi_x = 0. \tag{3.5}$$

The transverse curvature terms belonging on the right-hand sides of these equations are of order ϵ/r_0 or smaller and hence can be neglected to first order when the boundary-layer thickness is small compared with the cylinder radius. Their effect in the non-rotating case is reviewed by Van Dyke (1969). In what follows we shall take the ratio ϵ/r_0 to be small and neglect terms of that order in the boundary layer. Furthermore, the analysis is for the boundary layer forming on the outer surface of the cylinder ($y > 0$) only. The boundary layer on the inner surface will differ only to order ϵ/r_0.

The transformed boundary conditions are, then,

$$\Psi(x,0) = \Psi_\zeta(x,0) = 0 \quad (0 \leqslant x \leqslant 1), \tag{3.6a}$$

$$\Psi_\zeta(x,\infty) = 1, \quad V(x,\infty) = 0, \tag{3.6b}$$

and

$$V(x,0) = \delta/\sigma \,\mathrm{sgn}\,\Delta\Omega \quad (0 \leqslant x \leqslant 1). \tag{3.6c}$$

The scaling analysis which follows is presented in two sections for the cases $\delta \lesssim 1$ respectively. This division arises naturally as the nonlinear terms in the vorticity and momentum equations are always of smaller order of magnitude than the Coriolis terms and the flow is rotation-dominated when δ exceeds unity.

3.1. *Boundary-layer analysis for* $\delta < 1$

Two boundary-layer balances are possible for the vorticity equation (3.4) depending on the relative magnitudes of the various parameters. When advection and diffusion of vorticity form the primary vorticity balance, ϵ has the familiar form

$$\epsilon = R^{-\frac{1}{2}}, \tag{3.7}$$

with the further requirement that

$$\epsilon\sigma/Ro < 1. \tag{3.8}$$

Once σ is known, (3.8) defines the range for which this balance is applicable. The scaling parameter σ can be determined by examining the momentum equation (3.5) and the boundary condition (3.6*c*). Since the introduction of σ makes the transformed azimuthal velocity $V(x, \zeta)$ of order one, the boundary condition requires

$$\sigma \geqslant \sigma_1 = \delta. \tag{3.9}$$

However, when δ vanishes (i.e. no differential rotation between the cylinder and the ambient fluid), relative azimuthal motion must arise solely from the action of the Coriolis force through the coupling of the vorticity and momentum equations. The Coriolis and viscous terms in the momentum equation must then balance, leading to the condition

$$\sigma = \sigma_2 = \epsilon/Ro = 1/RoR^{\frac{1}{2}}. \tag{3.10}$$

Therefore, $\sigma = \sigma_2$ when the azimuthal velocity driven by the Coriolis force in the axial-flow boundary layer exceeds the azimuthal motion forced by the boundary condition (i.e. when $0 \leqslant \delta < (RoR^{\frac{1}{2}})^{-1}$) and $\sigma = \sigma_1$ when the opposite occurs (i.e. when $(RoR^{\frac{1}{2}})^{-1} < \delta < 1$). Returning to the requirement given in (3.8) we see that the advection–diffusion vorticity balance is characteristic of the side-wall boundary layer whenever

$$Ro > \epsilon = R^{-\frac{1}{2}}. \tag{3.11}$$

If $Ro < R^{-\frac{1}{2}}$ the characteristic balance must be that of the Coriolis and viscous terms or the entire flow is viscous-dominated.

Expanding the dependent variables in a perturbation sequence of the form

$$\psi(x,y) = \epsilon\Psi(x,\zeta) = \epsilon[\Psi^{(1)}(x,\delta) + \alpha(R, Ro, \delta)\,\Psi^{(2)}(x,\zeta) + \ldots] \tag{3.12a}$$

and
$$v(x,y) = \sigma V(x,\zeta) = \sigma[V^{(1)}(x,\zeta) + \phi(R, Ro, \delta)\, V^{(2)}(x,\zeta) + \ldots], \tag{3.12b}$$

and substituting into (3.4) and (3.5) yields the first-order boundary-layer equations. The stream function $\Psi^{(1)}$ satisfies the Blasius boundary-layer equation (cf. Rosenhead 1963, p. 222), and the azimuthal velocity $V^{(1)}$ satisfies the equations

$$[L(x,\zeta;\Psi^{(1)}) - \partial^2/\partial\zeta^2]\, V^{(1)} = 0, \quad V^{(1)}(x,0) = \operatorname{sgn}\Delta\Omega, \tag{3.13}$$

when $(RoR^{\frac{1}{2}})^{-1} < \delta < 1$ and satisfies the equation

$$\left[L(x,\zeta;\Psi^{(1)}) - \frac{\partial^2}{\partial\zeta^2}\right] V^{(1)} - 2\Psi^{(1)}_x = 0, \tag{3.14a}$$

when $0 \leqslant \delta \leqslant (RoR^{\frac{1}{2}})^{-1}$ with the wall boundary condition

$$V^{(1)}(x,0) = \begin{cases} 0, & \delta < (RoR^{\frac{1}{2}})^{-1}, \\ \operatorname{sgn}\Delta\Omega, & \delta = (RoR^{\frac{1}{2}})^{-1}. \end{cases} \tag{3.14b}$$

The solutions of these equations are presented later.

The above equations describe the first-order boundary-layer flow whenever $Ro > R^{-\frac{1}{2}}$. If $Ro < R^{-\frac{1}{2}}$, the Coriolis term in the vorticity equation (3.4) dominates the advective terms, and the primary boundary-layer balance is between the vorticity generated by Coriolis forces and that diffused by viscosity. The scaling parameter ϵ then becomes

$$\epsilon = (Ro/\sigma R)^{\frac{1}{3}}, \tag{3.15}$$

with the requirement

$$Ro/\sigma\epsilon < 1. \tag{3.16}$$

Following a procedure identical with that above reveals that the parameter σ is unity so long as $\delta \leqslant 1$. Thus we arrive at the familiar boundary-layer scaling

$$\epsilon = \epsilon_1 = (Ro/R)^{\frac{1}{3}} = E^{\frac{1}{3}}, \tag{3.17}$$

where E is the Ekman number ($E = \nu/\Omega_0 L^2$). If the expansions (3.12) are introduced, the first-order boundary-layer equations become precisely the Stewartson $\frac{1}{3}$-layer equations (cf. Greenspan 1968, p. 103):

$$\partial^4\Psi^{(1)}/\partial\zeta^4 - 2V^{(1)}_x = 0 \tag{3.18a}$$

and

$$V^{(1)}_{\zeta\zeta} + 2\Psi^{(1)}_x = 0. \tag{3.18b}$$

The azimuthal velocity satisfies the boundary condition

$$V^{(1)}(x,0) = \delta \operatorname{sgn}\Delta\Omega \quad (0 \leqslant x \leqslant 1). \tag{3.19}$$

Equations (3.18) are of sufficiently high order to satisfy all the requisite boundary conditions. The fact that $V^{(1)}(x,0)$ may be smaller than unity does not imply that $V^{(1)}$ is small everywhere and is not properly scaled. Within the boundary layer $V^{(1)}$ is of order unity because of the Coriolis force coupling of the vorticity and momentum equations. Thus, although the governing equations are identical to the familiar $E^{\frac{1}{3}}$-layer equations, the dynamics is quite different in that the axial-flow boundary layer induces an order-one (on this scale) azimuthal flow independent of the boundary conditions. The compatibility requirement (3.16) reveals that this single boundary layer is applicable whenever $Ro < E^{\frac{1}{2}}$ (or equivalently $Ro < R^{-\frac{1}{2}}$) and $\delta \leqslant 1$ (i.e. when the axial velocity is equal to or exceeds the difference in swirl velocities between the cylinder and the ambient fluid). When δ exceeds unity, a double-boundary-layer structure is required with the thinnest layer maintaining the $E^{\frac{1}{3}}$ scale.

Before discussing the case $\delta > 1$ we note that equations (3.18a,b) can be combined to yield a sixth-order parabolic equation for $V^{(1)}$ or $\Psi^{(1)}$ alone, each of which is invariant with respect to a reflexion of the x axis. The boundary layer is then blocked in the sense that an upstream wake appears. The parameter condition $Ro \gtrsim O(R^{-\frac{1}{2}})$ specifying the onset of blocking can be written in the form

$$Ro_{\delta_1} \sim U_0/\Omega_0\delta_1 \gtrsim O(1), \tag{3.20}$$

where δ_1 is the displacement thickness of the Blasius boundary layer. This is identical to the criterion derived by Kelly & Redekopp (1970) for the blocking of a stratified boundary layer on a horizontal surface if Ω_0 is replaced by the Brunt–Väisälä frequency N. Equation (3.20) implies that upstream influence occurs when the Rossby number based on the characteristic lateral dimension of an obstacle is of order one or smaller. This can be compared with the condition of $U_0/\Omega_0 a = 0{\cdot}77$ for the appearance of an upstream 'vortex bubble' reported by Orloff and Bossell (see Orloff 1971) in their experimental study of the axial flow of a rotating fluid over a transverse disk of radius a. Their result is obtained, however, from a linear extrapolation of their measured 'bubble' length as a function of the Rossby number. Maxworthy (1970) suggests that there is an upstream disturbance for any finite Rossby number, but it is not certain that this upstream influence differs significantly from that experienced by a potential flow when the Rossby number is greater than unity. Miles (1972) has shown theoretically that upstream separation occurs at a Rossby number of 1·05 in the inviscid flow of a rotating fluid over a disk. In the present case, the blocking condition is derived on the basis of a viscous analysis, but one can argue that the principal effect of viscosity, as far as blocking is concerned, is to give rise to a characteristic ateral dimension of the body.

3.2. *Boundary-layer analysis for $\delta > 1$*

When δ is much larger than unity, the azimuthal velocity boundary condition (3.6) dictates the choice $\sigma = \delta$. However, a Coriolis–viscous balance in both the vorticity and momentum equations can no longer hold simultaneously when $\sigma = \delta$. This is readily illustrated by noting that a Coriolis–viscous balance in the vorticity equation requires

$$\epsilon = (Ro/\sigma R)^{\frac{1}{3}}, \quad Ro/\sigma\epsilon < 1, \tag{3.21}$$

while a similar balance in the momentum equation yields

$$\epsilon = (\sigma Ro/R)^{\frac{1}{3}}, \quad \sigma Ro/\epsilon < 1. \tag{3.22}$$

These two expressions for ϵ, and also the corresponding compatibility relations, are satisfied simultaneously only when $\sigma = 1$, $\epsilon = \epsilon_1 = E^{\frac{1}{3}}$ and $Ro < E^{\frac{1}{3}}$. Clearly the boundary condition on the swirl velocity (which is of $O(\delta) > 1$) cannot be satisfied directly. The only alternative is to allow for a double structure in which a distinguished limiting set of equations describing motion in one layer yields solutions satisfying the azimuthal velocity conditions and another distinguished limiting set governing the motion in the other layer permits solutions satisfying the stream function conditions. Further, we insist that the solutions for the individual layers match uniformly in an overlap region, yielding a consistent dynamic description of the flow. This procedure yields the well-known $E^{\frac{1}{3}} - E^{\frac{1}{4}}$ sandwich boundary-layer structure as in contained rotating flows where the axial motion is due to Ekman-layer suction or blowing.

The only consistent boundary-layer scaling for which the stream function boundary conditions (i.e. for the axial and radial velocities) can be satisfied is that with $\epsilon = E^{\frac{1}{3}}$ and $\sigma = 1$ as derived above. Therefore, the $E^{\frac{1}{3}}$-layer must appear whenever $Ro < E^{\frac{1}{3}}$ (or, equivalently, $Ro < R^{-\frac{1}{2}}$), irrespective of the magnitude of δ, but it automatically allows for the satisfaction of the no-slip boundary condition on the azimuthal velocity only when $\delta \leqslant O(1)$. When δ exceeds unity an additional layer for which $\sigma = \delta$ and some of the boundary conditions on the stream function are relaxed is necessary. It is clear from (3.22) and from using the condition $\sigma = \delta > 1$ that the thickness of the second layer (ϵ_2, say) will be greater than ϵ_1.

A description of the flow in the outer layer can be derived in a straightforward manner by use of the transformations

$$\psi(x,y) = \beta\Psi_2(x,\zeta_2), \quad v(x,y) = \delta V_2(x,\zeta_2), \quad \zeta_2 = y/\epsilon_2. \tag{3.23}$$

In contrast to (3.2), the stream function ordering is left arbitrary since the boundary conditions on ψ must be relaxed. The scaling parameter β is determined subsequently by properly matching the velocities between the inner ϵ_1-layer and the outer ϵ_2-layer. Introducing the above transformations into the vorticity and momentum equations and neglecting transverse curvature terms yields, respectively,

$$\left[\frac{\beta}{\epsilon_2}L(x,\zeta_2;\Psi_2) - \frac{1}{\epsilon_2^2 R}\left(\epsilon_2^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \zeta_2^2}\right)\right]\left(\epsilon_2^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \zeta_2^2}\right)\Psi_2 + \frac{2\delta\epsilon_2}{\beta Ro}V_{2_x} = 0 \tag{3.24}$$

and

$$\left[\frac{\beta}{\epsilon_2}L(x,\zeta_2;\Psi_2) - \frac{1}{\epsilon_2^2 R}\left(\epsilon_2^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \zeta_2^2}\right)\right]V_2 - \frac{2\beta}{\delta Ro}\Psi_{2_x} = 0. \tag{3.25}$$

A Coriolis–viscous balance in the latter equation is necessary in order that the boundary conditions for V_2 can be satisfied. Thus,

$$\delta Ro/\epsilon_2^2\beta R = 1 \tag{3.26}$$

and, if β is known, ϵ_2 is determined. Now two ways of matching are possible for the determination of β: one in which the axial velocities match between the two boundary layers and the other in which the radial velocities match. The latter choice is the only one leading to a consistent perturbation problem. Hence, we must choose

$$\beta = \epsilon_1 = E^{\frac{1}{3}}, \tag{3.27}$$

whereby the value for ϵ_2 becomes

$$\epsilon_2 = (E^{\frac{2}{3}-n})^{\frac{1}{2}}. \tag{3.28}$$

In writing the above relation, we have replaced δ by n using the definition

$$\delta = E^{-n}. \tag{3.29}$$

Note that when $n = 0$ (i.e. $\delta = 1$) the outer layer reduces to thickness $E^{\frac{1}{3}}$ and as such is wholly contained within the ϵ_1 layer. For $\delta > 1$, the definition (3.29) implies that $n > 0$, while (3.28) reveals that $\epsilon_2 = 1$ and the entire flow is viscous if $n \geqslant \frac{2}{3}$. Thus, a double-boundary-layer structure always appears whenever δ lies in the range

$$1 < \delta < E^{-\frac{2}{3}}. \tag{3.30}$$

The first-order equations describing the motion in the ϵ_2 layer are

$$V_{2x}^{(1)} = 0 \tag{3.31}$$

and
$$\frac{\partial^2 V_2^{(1)}}{\partial \zeta_2^2} + 2\frac{\partial \Psi_2^{(1)}}{\partial x} = 0. \tag{3.32}$$

These equations are a weak form of the Taylor–Proudman theorem in that they require the azimuthal and radial velocities to be independent of the streamwise distance x while the axial velocity can vary at most linearly with x. They are identical to the Stewartson $E^{\frac{1}{4}}$-layer equations, but the boundary-layer scale is different.

Returning to (3.28), we note that the foregoing boundary-layer scaling reduces to the Stewartson $E^{\frac{1}{4}} - E^{\frac{1}{3}}$ double structure when $n = \frac{1}{6}$, that is, when the ratio of the axial velocity to the differential azimuthal velocity at the boundary of the sliced cylinder is of order $E^{\frac{1}{6}}$. By taking $n = \frac{1}{6}$ and computing the order of magnitude of the velocity components (when normalized with $\tilde{r}_0|\Delta\Omega|$ rather than U_0), it is a straightforward calculation to show that (u, v, w) are of order $(E^{\frac{1}{4}}, 1, E^{\frac{1}{2}})$ in the ϵ_2-layer and are of order $(E^{\frac{1}{6}}, E^{\frac{1}{6}}, E^{\frac{1}{2}})$ in the ϵ_1-layer $(\epsilon_1 = E^{\frac{1}{3}})$. These are precisely the scalings of the velocity components for the $E^{\frac{1}{3}} - E^{\frac{1}{4}}$ side-wall boundary-layer structure compatible with Ekman layers at the ends (Greenspan 1968). The preceding scaling analysis shows that the $E^{\frac{1}{3}}$-layer exists with the same dynamic equations whenever $Ro < E^{\frac{1}{6}}$, irrespective of the value of δ, and that an additional outer layer, similar dynamically to the $E^{\frac{1}{4}}$-layer, exists whenever $Ro < E^{\frac{1}{6}}$ and $1 < \delta < E^{-\frac{2}{3}}$, its thickness being a function of δ and given by $E^{\frac{2}{3}}\delta^{\frac{1}{2}}$.

4. A uniformly valid solution for $Ro > R^{-\frac{1}{2}}$

Similarity solutions are possible for the parameter range for which the advective and viscous diffusion terms comprise the primary vorticity balance in the boundary layer (i.e. when $Ro > R^{-\frac{1}{2}}$). The stream function $\Psi^{(1)}$ has the form

$$\Psi^{(1)}(x, \zeta) = x^{\frac{1}{2}} f(\eta), \quad \eta = \zeta / x^{\frac{1}{2}}, \tag{4.1}$$

where f satisfies the familiar Blasius equation. Asymptotically, the behaviour of f is given by

$$f \sim \eta - \delta_1, \tag{4.2}$$

where δ_1 is the displacement constant and equals 1·72. Equation (3.13) for the azimuthal velocity, applicable when $(RoR^{\frac{1}{2}})^{-1} < \delta < 1$, has a similarity solution given by the Pohlhausen equation (cf. Schlichting 1968, p. 280):

$$V^{(1)}(x, \zeta) = g(\eta),$$
$$g'' + \tfrac{1}{2} f g' = 0, \quad g(0) = \operatorname{sgn} \Delta\Omega, \quad g(\infty) = 0. \tag{4.3}$$

The solution of (3.14), applicable when $0 \leqslant \delta < (RoR^{\frac{1}{2}})^{-1}$, is more complicated. It has the similarity form

$$V^{(1)}(x, \zeta) = G_1(\eta) + x^{\frac{1}{2}} G_2(\eta), \tag{4.4}$$

where
$$G_1(\eta) = \begin{cases} 0, & \delta < (RoR^{\frac{1}{2}})^{-1}, \\ g(\eta) & \delta = (RoR^{\frac{1}{2}})^{-1}. \end{cases} \tag{4.5}$$

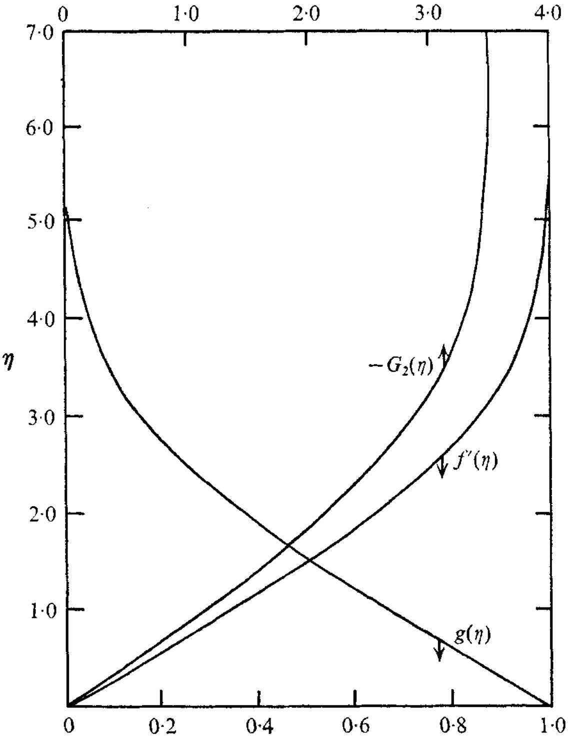

FIGURE 2. The axial and azimuthal boundary-layer profiles for $Ro > R^{-\frac{1}{2}}$ ($f''(0) = 0{\cdot}332$, $g'(0) = -0{\cdot}332 \operatorname{sgn} \Delta\Omega$, $G_2'(0) = -1{\cdot}086$).

The function $G_2(\eta)$ is determined by

$$\left.\begin{aligned} G_2'' + \tfrac{1}{2}fG_2' - \tfrac{1}{2}f'G_2 &= -(f-\eta f'), \\ G_2(0) = 0, \quad G_2(\infty) &= -2\delta_1. \end{aligned}\right\} \tag{4.6}$$

That the boundary condition $G_2(\infty)$ cannot be zero is readily seen by looking at the asymptotic solution for G_2. The functions f, g and G_2 are plotted in figure 2, from which the axial and azimuthal velocity components in the boundary-layer region can be computed.

The above solutions are not uniformly valid since the radial velocity and the azimuthal velocity from (4.6) are not zero outside the boundary layer. To correct this, another distinguished limit of the equations must be found and must yield solutions which match the boundary-layer variables $\Psi^{(1)}$ and $V^{(1)}$ and also satisfy the condition of uniform axial motion and zero swirl as $r \to \infty$. The correct limit can be shown to consist of a balance of the advective and Coriolis terms in both the vorticity and momentum equations.

In order to simplify the mathematical features of the matching and to emphasize the structure of the outer flow, curvature effects will be neglected ($r_0 \gg 1$). The solution to the momentum equation (2.8) is then

$$v(x,y) = (2/Ro)\,(\psi(x,y) - y), \tag{4.7}$$

since viscous effects are of higher order. The correct outer expansions are

$$\psi(x,y) = y + \gamma(R, Ro)\,\psi^{(1)} + \ldots \tag{4.8a}$$

and $$v(x,y) = \frac{1}{Ro}[0+\gamma(R,Ro)\,v^{(1)}+\ldots], \tag{4.8b}$$

which lead to the first-order equations

$$\nabla_1^2\psi^{(1)}+4R^{-2c}\psi^{(1)} = 0, \quad v^{(1)} = 2\psi^{(1)}. \tag{4.9}$$

The substitution $Ro = R^c$ has been made for convenience. Recalling that the foregoing boundary-layer solutions are applicable for $Ro > R^{-\frac{1}{2}}$, we obtain $c > -\frac{1}{2}$. The matching of the stream function requires

$$\gamma(R,Ro)\,\psi^{(1)}(x,0) = -\epsilon\,\delta_1 x^{\frac{1}{2}}, \tag{4.10}$$

whereby $\gamma = \epsilon = R^{-\frac{1}{2}}$. The matching requirement on the azimuthal velocity from (4.4) and (4.6) is automatically satisfied and the problem for $\psi^{(1)}$ is well posed when $c > 0$ (i.e. $Ro \geqslant O(1)$). However, when

$$-\tfrac{1}{2} < c < 0 \quad (R^{-\frac{1}{2}} < Ro < O(1))$$

the equation for $\psi^{(1)}$ becomes, in the limit of large Reynolds number,

$$\partial\psi^{(1)}/\partial x = 0, \tag{4.11}$$

which clearly does not admit a solution satisfying the matching condition (4.10).

The same problem was encountered by Kelly & Redekopp (1970) in their study of the boundary layer on a flat plate in a stratified flow. They showed that the correct description of the flow could be obtained by introducing an intermediate layer in which both independent variables are rescaled. With the rescaled variables denoted by $\hat{x}$ and $\hat{y}$, the flow in the intermediate layer (valid for $-\frac{1}{2} < c < 0$) is represented by

$$(\hat{x},\hat{y}) = (x,y)/R^c, \tag{4.12a}$$

$$\psi(x,y) = R^c[\hat{y}+R^{-\frac{1}{2}(1+c)}\hat{\Psi}^{(1)}(\hat{x},\hat{y})+\ldots] \tag{4.12b}$$

and $$v(x,y) = R^{-\frac{1}{2}(1+c)}\hat{V}^{(1)}(\hat{x},\hat{y})+\ldots, \tag{4.12c}$$

with the governing equations

$$(\hat{\nabla}_1^2+4)\hat{\Psi} = 0, \quad \hat{V}^{(1)} = 2\hat{\Psi}^{(1)} \tag{4.13a}$$

and matching condition

$$\hat{\Psi}^{(1)}(\hat{x},0) = -\delta_1 x^{\frac{1}{2}}. \tag{4.13b}$$

The swirl velocity is again properly matched, showing that it decays to zero in an outer inviscid layer. The solutions for $\psi^{(1)}$ and $\hat{\Psi}^{(1)}$ are discussed in Kelly & Redekopp (1970, §6). The scaling (4.12*a*) for the independent variables is equivalent to non-dimensionalizing the co-ordinate lengths with the wavelength U_0/Ω_0 of inertial waves moving with phase velocity U_0 and oscillating at the natural frequency Ω_0.

5. Summary

The foregoing analysis has shown how the boundary-layer structure changes from the Blasius type to the Stewartson type as the axial velocity varies in relation to azimuthal velocities in a rotating fluid. No other distinct viscous boundary layers appear as the flow changes from one type to the other except that the thickness of the familiar $\frac{1}{4}$-layer varies according to the relative magnitudes of these velocities. The boundary-layer flow is found to exhibit upstream

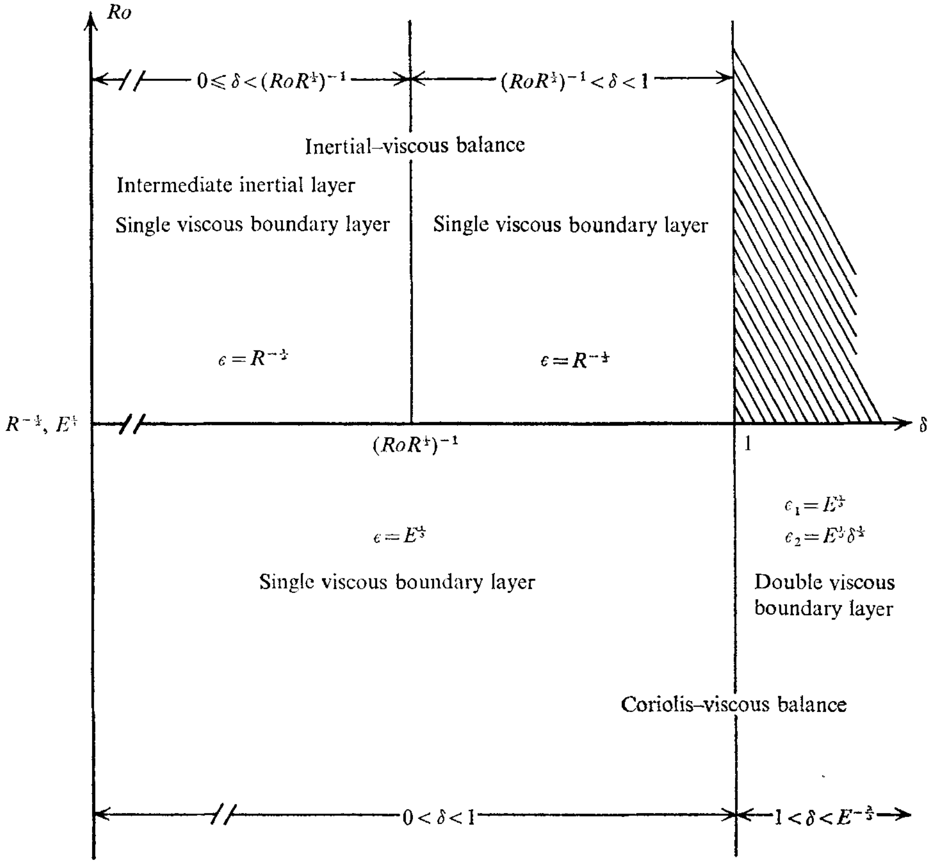

FIGURE 3. A representation of the important transition regions in (Ro, δ) parameter space.

influence whenever $Ro \gtrsim O(R^{\frac{1}{2}})$ or, equivalently, $Ro \gtrsim O(E^{\frac{1}{2}})$. The flow structure is summarized graphically in figure 3 in terms of the three parameters Ro, R (or E) and δ.

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